

Extrapolation from $A_\infty^{\rho,\infty}$, vector-valued inequalities and applications in the Schrödinger settings

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Abstract In this paper, we generalize the A_∞ extrapolation theorem in [6] and the A_p extrapolation theorem of Rubio de Francia to Schrödinger settings. In addition, we also establish the weighted vector-valued inequalities for Schrödinger type maximal operators by using weights belonging to $A_p^{\rho,\theta}$ which includes A_p . As their applications, we establish the weighted vector-valued inequalities for some Schrödinger type operators and pseudo-differential operators.

1. Introduction

In this paper, we consider the Schrödinger differential operator $L = -\Delta + V(x)$ on \mathbb{R}^n , $n \geq 3$, where $V(x)$ is a nonnegative potential satisfying certain reverse Hölder class.

We say a nonnegative locally L^q integral function $V(x)$ on \mathbb{R}^n is said to belong to $B_q(1 < q \leq \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y) dy \right)^{1/q} \leq C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V(y) dy \right)$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r . In particular, if V is a nonnegative polynomial, then $V \in B_\infty$. Throughout this paper, we always assume that $0 \not\equiv V \in B_n/2$.

The study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention; see [4, 5, 7, 8, 20, 28]. In particular, it should be pointed out that Shen [20] proved the Schrödinger type operators, such as $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$, $(-\Delta + V)^{-1/2}\nabla$ with $V \in B_n$, $(-\Delta + V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, are standard Calderón-Zygmund operators.

Recently, Bongioanni, etc, [4] proved $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) boundedness for commutators of Riesz transforms associated with Schrödinger operator with $BMO_\theta(\rho)$ functions which include the class BMO function, and in [5] established the weighted boundedness for Riesz transforms, fractional integrals and Littlewood-Paley functions associated with

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Schrödinger operator with weight $A_p^{\rho, \theta}$ class which includes the Muckenhoupt weight class. Very recently, the author [23, 24] established the weighted norm inequalities for some Schrödinger type operators, which include commutators of Riesz transforms, fractional integrals and Littlewood-paley operators.

On the other hand, extrapolation for weights plays an important role in Harmonic analysis. In particular, Rubio de Francia [19] proved the A_p extrapolation theorem: If the operator is bounded on $L^{p_0}(\omega)$ for some p_0 , $1 < p_0 < \infty$, and every $\omega \in A_{p_0}$, then for every p , $1 < p < \infty$, T is bounded on $L^p(\omega)$, $\omega \in A_p$ (see also [9, 13]). Recently, Cruz-Uribe, etc, in [6] extended this theorem from A_p weights to A_∞ weights, to pairs of operators, and to the range $0 < p < \infty$ in the context of Muckenhoupt bases.

In this paper, we generalize the A_∞ extrapolation theorem in [6] and the A_p extrapolation theorem of Rubio de Francia to Schrödinger settings and give some applications.

The paper is organized as follows. In Section 2, we give factorization of $A_p^{\rho, \infty}$, and establish the weighted vector-valued inequalities for Schrödinger type maximal operators, these results play a crucial role in this paper. In Section 3, we obtain extrapolation theorems from $A_\infty^{\rho, \infty}$ and $A_p^{\rho, \infty}$. Finally, we establish the weighted vector-valued inequalities for some Schrödinger type operators and pseudo-differential operators in section 4.

Throughout this paper, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Factorization and vector-valued inequalities

In this section, we give the factorization of $A_p^{\rho, \infty}$ and weighted vector-valued inequalities for Schrödinger type maximal operators.

We first recall some notation. Given $B = B(x, r)$ and $\lambda > 0$, we will write λB for the λ -dilate ball, which is the ball with the same center x and with radius λr . Similarly, $Q(x, r)$ denotes the cube centered at x with the sidelength r (here and below only cubes with sides parallel to the coordinate axes are considered), and $\lambda Q(x, r) = Q(x, \lambda r)$. Let $f = \{f_k\}_1^\infty$ is a sequence of locally integral functions \mathbb{R}^n , $|f(x)|_r = (\sum_{k=1}^\infty |f_k(x)|^r)^{1/r}$, and $|Tf(x)|_r = (\sum_{k=1}^\infty |Tf_k(x)|^r)^{1/r}$.

The function $m_V(x)$ is defined by

$$\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim (1 + |x|)$ with $V = |x|^2$.

Lemma 2.1([20]). *There exists $l_0 > 0$ and $C_0 > 1$ such that*

$$\frac{1}{C_0} (1 + |x - y| m_V(x))^{-l_0} \leq \frac{m_V(x)}{m_V(y)} \leq C_0 (1 + |x - y| m_V(x))^{l_0/(l_0+1)}.$$

In particular, $m_V(x) \sim m_V(y)$ if $|x - y| < C/m_V(x)$.

In this paper, we write $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, where $\theta > 0$, x_0 and r denotes the center and radius of B respectively.

A weight will always mean a positive function which is locally integrable. As [5], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ for $1 < p < \infty$, if there is a constant C such that for all balls B

$$\left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega(y) dy \right) \left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1} \leq C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C such that

$$M_{V,\theta}(\omega)(x) \leq C\omega(x), \text{ a.e. } x \in \mathbb{R}^n.$$

where

$$M_{V,\theta}f(x) = \sup_{x \in B} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(y)| dy.$$

When $V = 0$, we denote $M_0f(x)$ by $Mf(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \leq M_{V,\theta}f(x) \leq Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and any $\theta \geq 0$.

Since $\Psi_\theta(B) \geq 1$ with $\theta \geq 0$, then $A_p \subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$, where A_p denotes the classical Muckenhoupt weights; see [14] and [16]. We will see that $A_p \subset\subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$ in some cases. In fact, let $\theta > 0$ and $0 \leq \gamma \leq \theta$, it is easy to check that $\omega(x) = (1 + |x|)^{-(n+\gamma)} \notin A_\infty$ and $\omega(x)dx$ is not a doubling measure, but $\omega(x) = (1 + |x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$ provided that $V = 1$ and $\Psi_\theta(B(x_0, r)) = (1 + r)^\theta$.

We remark that balls can be replaced by cubes in definition of $A_p^{\rho,\theta}$ and $M_{V,\theta}$, since $\Psi(B) \leq \Psi(2B) \leq 2^\theta \Psi(B)$.

Next we give the weighted boundedness of $M_{V,\theta}$.

Lemma 2.2([22]). *Let $1 < p < \infty$, $p' = p/(p-1)$ and assume that $\omega \in A_p^{\rho,\theta}$. There exists a constant $C > 0$ such that*

$$\|M_{V,p'\theta}f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Similar to the classical Muckenhoupt weights(see [15, 14, 21]), we give some properties for weight class $A_p^{\rho,\theta}$ for $p \geq 1$.

Proposition 2.1. *Let $\omega \in A_p^{\rho,\infty} = \bigcup_{\theta \geq 0} A_p^{\rho,\theta}$ for $p \geq 1$. Then*

- (i) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\rho,\theta} \subset A_{p_2}^{\rho,\theta}$.*
- (ii) *$\omega \in A_p^{\rho,\theta}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^{\rho,\theta}$, where $1/p + 1/p' = 1$.*
- (iii) *If $\omega \in A_p^{\rho,\infty}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^{\rho,\infty}$.*
- (vi) *Let $f \in L_{loc}(\mathbb{R}^n)$, $0 < \delta < 1$, then $(M_{V,\theta})^\delta \in A_1^{\rho,\theta}$.*
- (v) *Let $1 < p < \infty$, then $\omega \in A_p^{\rho,\infty}$ if and only if $\omega = \omega_1\omega_2^{1-p}$, where $\omega_1, \omega_2 \in A_1^{\rho,\infty}$.*

Proof. (i) and (ii) are obvious by the definition of $A_p^{\rho, \theta}$. (iii) is proved in [5]. In fact, from Lemma 5 in [5], we know that if $\omega \in A_p^{\rho, \theta}$, then $\omega \in A_{p_0}^{\rho, \theta_0}$, where $p_0 = 1 + \frac{p-1}{1-\delta} < p$ with $0 < \delta < 1$ (δ is a constant depending only on the $A_p^{\rho, loc}$ constant of ω , see [5]) and $\theta_0 = \frac{\theta p + \eta(p-1)}{p_0}$ with $\eta = \theta p + (\theta + n)\frac{p l_0}{l_0 + 1} + (l_0 + 1)\frac{n\delta}{1+\delta}$. We now prove (vi). It will suffice to show that there exists a constant C such that for every f , every cube Q and almost every $x \in Q$,

$$\frac{1}{\Psi_\theta(Q)|Q|} \int_Q M_{V, \theta} f(y)^\delta dy \leq C M_{V, \theta} f(x)^\delta.$$

Fix Q and decompose f as $f = f_1 + f_2$, where $f_1 = f \chi_{2Q}$. Then $M_{V, \theta} f(x) \leq M_{V, \theta} f_1(x) + M_{V, \theta} f_2(x)$, and so for $0 \leq \delta < 1$,

$$M_{V, \theta} f(x)^\delta \leq M_{V, \theta} f_1(x)^\delta + M_{V, \theta} f_2(x)^\delta.$$

Since $M_{V, \theta}$ is weak $(1, 1)$, by Kolmogorev's inequality (see [18])

$$\begin{aligned} \frac{1}{\Psi_\theta(Q)|Q|} \int_Q (M_{V, \theta} f_1)^\delta(y) dy &\leq \frac{C}{\Psi_\theta(Q)|Q|} |Q|^{1-\delta} \|f_1\|_1^\delta \\ &\leq C \left(\frac{1}{\Psi_\theta(Q)|Q|} \int_{2Q} |f(y)| dy \right)^\delta \\ &\leq C \left(\frac{1}{\Psi_\theta(2Q)|2Q|} \int_{2Q} |f(y)| dy \right)^\delta \\ &\leq C M_{V, \theta} f(x)^\delta. \end{aligned}$$

To estimate $M_{V, \theta} f_2$, note that let Q' is a cube such that $x \in Q'$, if $Q' \cap (\mathbb{R}^n \setminus (2Q)) \neq \emptyset$, then $Q \subset 4nQ'$. Hence, for any $z \in Q$

$$\frac{1}{\Psi_\theta(Q')|Q'|} \int_{Q'} |f_2(y)| dy \leq \frac{C}{\Psi_\theta(4nQ')|4nQ'|} \int_{4nQ'} |f_2(y)| dy \leq C M_{V, \theta}(z).$$

So $M_{V, \theta}(y) \leq C M_{V, \theta}(x)$ for any $y \in Q$. Thus

$$\frac{1}{\Psi_\theta(Q')|Q'|} \int_{Q'} M_{V, \theta} f_2(y)^\delta dy \leq C M_{V, \theta} f(x)^\delta.$$

It remains to prove (v). We first assume $\omega_1 \in A_1^{\rho, \theta_1}$ and $\omega_2 \in A_1^{\rho, \theta_2}$. Since

$$\begin{aligned} \left(\frac{1}{\Psi_{\theta_1}(Q)|Q|} \int_Q \omega_1(y) dy \right) \left(\inf_Q \omega_1(y) \right)^{-1} &\leq C_1, \\ \left(\frac{1}{\Psi_{\theta_2}(Q)|Q|} \int_Q \omega_2(y) dy \right) \left(\inf_Q \omega_2(y) \right)^{-1} &\leq C_2, \end{aligned}$$

moreover

$$\begin{aligned} \frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega(y) dy &= \frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega_1(y) \omega_2^{1-p}(y) dy \\ &\leq \left(\frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega_1(y) dy \right) \left(\inf_Q \omega_2(y) \right)^{1-p}, \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1} &= \left(\frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega_1^{-\frac{1}{p-1}}(y) \omega_2(y) dy \right)^{p-1} \\ &\leq \left(\frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega_2(y) dy \right)^{p-1} \left(\inf_Q \omega_1(y) \right)^{-1}. \end{aligned}$$

From these inequalities above and choosing $\theta = \max\{\theta_1, \theta_2\}$, then

$$\left(\frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega(y) dy \right) \left(\frac{1}{\Psi_\theta(Q)|Q|} \int_Q \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1} \leq C_1 C_2^{p-1}.$$

To prove the converse, we consider first $p \geq 2$, let $\omega \in A_p^{\rho, \theta}$, and define T by

$$Tf = [\omega^{-1/p} M_{V, p\theta}(f^{p/p'} \omega^{1/p})]^{p'/p} + \omega^{1/p} M_{V, p\theta}(f \omega^{-1/p}).$$

Because $\omega^{-p'/p} \in A_{p'}^{\rho, \theta}$, then T is bounded on L^p by Lemma 2.2, that is,

$$\|Tf\|_{L^p} \leq A\|f\|_{L^p},$$

for some $A > 0$. Also, since $p \geq 2, p/p' \geq 1$, and Minkowski's inequality gives $T(f_1 + f_2) \leq Tf_1 + Tf_2$. Fix now a nonnegative f with $\|f\|_{L^p} = 1$ and write

$$\eta = \sum_{k=1}^{\infty} (2A)^{-k} T^k(f),$$

where $T^k(f) = T(T^{k-1}(f))$. Then $\|\eta\|_{L^p} \leq 1$. Furthermore, since T is positivity-preserving and subadditive, we have the pointwise inequality

$$T\eta \leq \sum_{k=1}^{\infty} (2A)^{-k} T^{k+1}(f) = \sum_{k=2}^{\infty} (2A)^{1-k} T^k(f) \leq 2A\eta.$$

Thus, if $\omega_1 = \omega^{1/p} \eta^{p/p'}$, then

$$M_{V, p\theta}(\omega_1) \leq (T(\eta))^{p/p'} \omega^{1/p} \leq (2A\eta)^{p/p'} \omega^{1/p} = (2A)^{p/p'} \omega_1$$

and $\omega \in A_1^{\rho, p\theta}$. Similarly, if $\omega_2 = \omega^{-1/p} \eta$, then $M_{V, p\theta}(\omega_1) \leq 2A\omega_2$, so $\omega_2 \in A_1^{\rho, p\theta}$. Moreover,

$$\omega = \omega_1 \omega_2^{1-p} = \omega^{1/p} \eta^{p/p'} (\omega^{-1/p} \eta)^{1-p},$$

since $p/p' = p - 1$, finishing the proof or $p \geq 2$.

The case $p \leq 2$ is similar. In fact, let $\omega \in A_p^{\rho, \theta}$, then $\omega^{-p'/p} \in A_{p'}^{\rho, \theta}$, and define T by

$$Tf = [\omega^{1/p} M_{V, p'\theta}(f^{p'/p} \omega^{-1/p})]^{p/p'} + \omega^{-1/p} M_{V, p'\theta}(f \omega^{1/p}).$$

then T is bounded on L^p by Lemma 2.2, that is,

$$\|Tf\|_{L^{p'}} \leq B\|f\|_{L^p},$$

for some $A > 0$. Also, since $p \leq 2, p'/p \geq 1$, and Minkowski's inequality gives $T(f_1 + f_2) \leq Tf_1 + Tf_2$. Fix now a nonnegative f with $\|f\|_{L^{p'}} = 1$ and write

$$\eta = \sum_{k=1}^{\infty} (2B)^{-k} T^k(f),$$

where $T^k(f) = T(T^{k-1}(f))$. Then $\|\eta\|_{L^{p'}} \leq 1$. Furthermore, since T is positivity-preserving and subadditive, we have the pointwise inequality

$$T\eta \leq \sum_{k=1}^{\infty} (2B)^{-k} T^{k+1}(f) = \sum_{k=2}^{\infty} (2B)^{1-k} T^k(f) \leq 2B\eta.$$

Thus, if $\omega_1 = \omega^{-1/p} \eta^{p'/p}$, then

$$M_{V,p\theta}(\omega_1) \leq (T(\eta))^{p'/p} \omega^{-1/p} \leq (2B\eta)^{p'/p} \omega^{1/p} = (2B)^{p'/p} \omega_1$$

and $\omega \in A_1^{\rho,p'\theta}$. Similarly, if $\omega_2 = \omega^{1/p} \eta$, then $M_{V,p'\theta}(\omega_1) \leq 2B\omega_2$, so $\omega_2 \in A_1^{\rho,p'\theta}$. Moreover,

$$\omega = \omega_2 \omega_1^{1-p} = \omega^{1/p} \eta (\omega^{-1/p} \eta^{p'/p})^{1-p},$$

since $p/p' = p - 1$, finishing the proof for $p \leq 2$. The proof is complete. \square

C. Fefferman and E. Stein [10] obtained the vector-valued inequalities of Hardy-Littlewood maximal operators. Later, K. Andersen and R. John [1] generalized the Fefferman-Stein vector-valued inequalities to A_p weights case. We next give some weighted vector-valued inequalities of maximal operators $M_{V,\eta}$ by new weights. The following interpolation results will be required. Let \mathcal{S} denote the linear space of sequence $f = \{f_k\}$ of the form: $f_k(x)$ is a simple function on \mathbb{R}^n and $f_k(x) \equiv 0$ for all sufficient large k . \mathcal{S} is dense in $L_\omega^p(l^r)$, $1 \leq p, r < \infty$; see [2].

Lemma 2.3([1]). *Let $\omega \geq 0$ be locally integral on \mathbb{R}^n , $1 < r < \infty$, $1 \leq p_i \leq q_i < \infty$ and suppose T is a sublinear operator defined on \mathcal{S} satisfying*

$$\omega(\{x \in \mathbb{R}^n : |Tf(x)|_r > \alpha\}) \leq M_i^{q_i} \alpha^{-q_i} \left(\int_{\mathbb{R}^n} |f(x)|_r^{p_i} \omega(x) dx \right)^{q_i/p_i}$$

for $i = 0, 1$ and $f \in \mathcal{S}$. Then T extends uniquely to a sublinear operator on $L_\omega^p(l^r)$ and there is a constant M_θ such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|_r^q \omega(x) dx \right)^{1/q} \leq M_\theta \left(\int_{\mathbb{R}^n} |f(x)|_r^{p_i} \omega(x) dx \right)^{1/p}$$

where $(1/p, 1/q) = (1 - \theta)(1/p_0, 1/q_0) + \theta(1/p_1, 1/q_1)$, $0 < \theta < 1$.

Lemma 2.4([1]). *Let $\omega \geq 0$ be locally integral on \mathbb{R}^n , $1 < r_i, s_i < \infty$, $1 \leq p_i, q_i < \infty$ and suppose T is a sublinear operator defined on \mathcal{S} satisfying*

$$\left(\int_{\mathbb{R}^n} |Tf(x)|_{s_i}^{q_i} \omega(x) dx \right)^{1/q_i} \leq M_i \left(\int_{\mathbb{R}^n} |f(x)|_{r_i}^{p_i} \omega(x) dx \right)^{1/p_i}$$

for $i = 0, 1$ and $f \in \mathcal{S}$. Then T extends uniquely to a sublinear operator on $L_\omega^p(l^r)$ and there is a constant M_θ such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|_r^q \omega(x) dx \right)^{1/q} \leq M_0^{1-\theta} M_1^\theta \left(\int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx \right)^{1/p}$$

where $(1/p, 1/q, 1/s, 1/r) = (1-\theta)(1/p_0, 1/q_0, 1/s_0, 1/r_0) + \theta(1/p_1, 1/q_1, 1/s_1, 1/r_1)$, $0 < \theta < 1$.

We define the dyadic maximal operator $M_{V,\theta}^\Delta f(x)$ as follows

$$M_{V,\theta}^\Delta f(x) := \sup_{x \in Q(\text{dyadic cube})} \frac{1}{\psi_\theta(Q)|Q|} \int_Q |f(x)| dx,$$

where $\psi_\theta(Q) = (1 + r/\max_Q \rho(x))^\theta$, r is side-length of Q and $\theta > 0$.

Lemma 2.5. *Let f be a locally integrable function on \mathbb{R}^n , $\lambda > 0$, and $\Omega_\lambda = \{x \in \mathbb{R}^n : M_{V,\theta}^\Delta f(x) > \lambda\}$. Then Ω_λ may be written as a disjoint union of dyadic cubes $\{Q_j\}$ with*

$$(i) \quad \lambda < (\psi_\theta(Q_j)|Q_j|)^{-1} \int_{Q_j} |f(x)| dx,$$

$$(ii) \quad (\psi_\theta(Q_j)|Q_j|)^{-1} \int_{Q_j} |f(x)| dx \leq (4n)^\theta 2^n \lambda, \text{ for each cube } Q_j. \text{ This has the immediate consequences:}$$

$$(iii) \quad |f(x)| \leq \lambda \text{ for a.e } x \in \mathbb{R}^n \setminus \bigcup_j Q_j$$

$$(iv) \quad |\Omega_\lambda| \leq \lambda^{-1} \int_{\mathbb{R}^n} |f(x)| dx.$$

The proof follows from the same argument of Lemma 1 in page 150 of [21].

Theorem 2.1. *Let $1 < r < \infty$ and $\theta > 0$.*

(a) *If $1 \leq p < \infty$, $\omega \in A_p^{\rho,\theta}$, $\eta = p_0 \theta_0$ where $p_0 = 4(l_0 + 1)^5(p + (\frac{r+1}{2})')$ and $\theta_0 = p((3\theta + n)p + (l_0 + 1)n)$, there is a constant C_{r,p,θ,l_0,C_0} such that*

$$\omega(\{x \in \mathbb{R}^n : |M_{V,\eta} f(x)|_r > \alpha\}) \leq C \alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx. \quad (2.1)$$

(b) *If $1 < p < \infty$, $\omega \in A_p^{\rho,\theta}$ and η be same as above, there is a constant C_{r,p,θ,l_0,C_0} such that*

$$\int_{\mathbb{R}^n} |M_{V,\eta} f(x)|_r^p \omega(x) dx \leq C \alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx. \quad (2.2)$$

Proof. Observe first that (2.2) for the case $r = p$ is easy consequence of Lemma 2.2 since $\eta > r'\theta$,

$$\begin{aligned} \int_{\mathbb{R}^n} |M_{V,\eta} f(x)|_r^r \omega(x) dx &= \sum_k \int_{\mathbb{R}^n} |M_{V,\eta} f_k(x)|^r \omega(x) dx \\ &\leq C \sum_k \int_{\mathbb{R}^n} |f_k(x)|^r \omega(x) dx \\ &= C \sum_k \int_{\mathbb{R}^n} |f_k(x)|_r^r \omega(x) dx. \end{aligned} \quad (2.3)$$

Now suppose $r > p$, $\omega \in A_p^{\rho,\theta}$ and $\alpha > 0$. As usual, we can assume that $f \in C_0^\infty$. Let $\theta_1 = \theta(l_0 + 1)$. From Lemma 2.5, we yields a sequence of non-overlapping cube $\{Q_j\}$ such that

$$|f(x)|_r \leq \alpha, \quad x \notin \Omega = \bigcup_{j=1}^{\infty} Q_j, \quad (2.4)$$

$$\alpha < \frac{1}{\psi_{\theta_1}(Q_j)|Q_j|} \int_{Q_j} |f(x)|_r dx \leq 2^n (4n)^{\theta_1} \alpha, \quad j = 1, 2, \dots. \quad (2.5)$$

Let $f = f' + f''$ where $f' = \{f'_k\}$, $f'_k(x) = f_k(x) \chi_{\mathbb{R}^n \setminus \Omega}(x)$. Then

$$|M_{V,\eta} f(x)|_r \leq |M_{V,\eta} f'(x)|_r + |M_{V,\eta} f''(x)|_r.$$

From this, (2.1) will follow if we show that

$$\omega(\{x \in \mathbb{R}^n : |M_{V,\eta} f'(x)|_r > \alpha\}) \leq C \alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx. \quad (2.6)$$

and

$$\omega(\{x \in \mathbb{R}^n : |M_{V,\eta} f''(x)|_r > \alpha\}) \leq C \alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx. \quad (2.7)$$

Since $\omega \in A_r^{\rho,\theta}$ by (i) of Proposition 2.1, from (2.3) and (2.4), we then have

$$\omega(\{x \in \mathbb{R}^n : |M_{V,\eta} f'(x)|_r > \alpha\}) \leq C \alpha^{-r} \int_{\mathbb{R}^n} |f(x)|_r^r \omega(x) dx \leq C \alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx.$$

Thus, (2.6) is proved. To prove (2.7), define $\bar{f} = \{\bar{f}_k\}$ by

$$\bar{f}_k(x) = \frac{1}{\psi_{\theta_1}(Q_j)|Q_j|} \int_{Q_j} |f_k(y)| dy, \quad x \in Q_j, \quad j = 1, 2, \dots,$$

zero, otherwise. Let $\bar{Q}_j = 2nQ_j$. We now claim that for any $x \in \bar{\Omega} = \bigcup_j \bar{Q}_j$,

$$M_{V,\eta} f''_k(x) \leq C M_{V,\bar{\eta}} \bar{f}_k(x), \quad \forall k,$$

where $\bar{\eta} = \eta/2(l_0 + 1)^2$.

In fact, $\forall x \notin \bar{\Omega}$, and cube $Q \ni x$, if $Q_j \cap Q \neq \emptyset$, then $Q_j \subset \bar{Q} = 4nQ$, hence

$$\begin{aligned}
 \frac{1}{\Psi_\eta(Q)|Q|} \int_Q |f_k''(x)| dx &= \frac{1}{\Psi_\eta(Q)|Q|} \sum_j \int_{Q_j \cap Q} |f_k(x)| dx \\
 &\leq \frac{1}{\Psi_\eta(Q)|Q|} \sum_{Q_j \subset \bar{Q}} \int_{Q_j} |f_k(x)| dx \\
 &\leq \frac{1}{\Psi_\eta(Q)|Q|} \sum_{Q_j \subset \bar{Q}} \psi_{\theta_1}(Q_j) \int_{Q_j} \bar{f}_k(x) dx \\
 &\leq C \frac{\Psi_{\theta_2}(\bar{Q})}{\Psi_\eta(Q)|Q|} \int_{\bar{Q}} \bar{f}_k(x) dx \\
 &\leq CM_{V, \bar{\eta}} \bar{f}_k(x),
 \end{aligned}$$

where $\theta_2 = \theta_1(l_0 + 1) = \theta(l_0 + 1)^2$.

By the claim above, it is easy to see that (3.8) will follow if we show

$$\omega(\bar{\Omega}) \leq C\alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx. \quad (2.8)$$

and

$$\omega(\{x \in \mathbb{R}^n : |M_{V, \bar{\eta}} \bar{f}(x)|_r > \alpha\}) \leq C\alpha^{-p} \int_{\mathbb{R}^n} |f(x)|_r^p \omega(x) dx. \quad (2.9)$$

If $p > 1$, by (2.5), we then have

$$\begin{aligned}
 \omega(\bar{Q}_j) &= \int_{\bar{Q}_j} \omega(x) dx \leq \frac{\alpha^{-p}}{(\psi_{\theta_1}(Q)|Q|)^p} \left(\int_{Q_j} |f(x)|_r \right)^p \int_{\bar{Q}_j} \omega(x) dx \\
 &\leq \alpha^{-p} \left(\int_{Q_j} |f(x)|_r^p \omega(x) dx \right) \left(\frac{1}{(\Psi_\theta(Q)|Q|)} \int_{Q_j} \omega^{-1/(p-1)}(x) dx \right)^{p-1} \\
 &\quad \times \left(\frac{1}{(\Psi_\theta(Q)|Q|)} \int_{\bar{Q}_j} \omega(x) dx \right) \\
 &\leq \alpha^{-p} \int_{Q_j} |f(x)|_r^p \omega(x) dx,
 \end{aligned} \quad (2.10)$$

since $\omega \in A_p^{\rho, \theta}$.

A similar argument shows that (2.10) holds also if $p = 1$. Hence, (2.8) follows from (2.10) upon summing over j . Note that $|\bar{f}(x)|_r \leq 2^n(4n)^{\theta_1} \alpha$, and since $|\bar{f}(x)|_r$ is supported in Ω , using Lemma 2.2, we obtain

$$\omega(\{x \in \mathbb{R}^n : |M_{V, \bar{\eta}} \bar{f}(x)|_r > \alpha\}) \leq C\alpha^{-r} \int_{\mathbb{R}^n} |\bar{f}(x)|_r^r \omega(x) dx \leq C \int_{\Omega} \omega(x) dx$$

which together with (2.10) yields (2.9) as required. This complete the proof (2.1) in the case $r \geq p$. If $r > p > 1$, by (iii) of Proposition 2.1, we know that for $\omega \in A_p^{\rho, \theta}$, there exist constants p_1, p_2, θ_3 (depending only on ω) $(r+1)/2 < p_1 < p < p_2 < r$ and $\theta_3 \leq \theta_0$ so that

(2.1) holds with $\omega \in A_{p_1}^{\theta_3}$ and $\omega \in A_{p_2}^{\theta}$ respectively. Obviously, $\bar{\eta} > 2p'_1\theta_3$, Lemmas 2.2 and 2.3 yields (2.2) for $r > p > 1$.

Suppose now that $p > r$ and $\omega \in A_p^{\rho, \theta}$. By (iii) of Proposition 2.1, there exist constants $\theta_4 \leq \theta_0$ and $1 < r_0 < p$ such that $\omega \in A_q^{\rho, \theta_4}$, $q \geq p/r_0$. In particular, (i) of Proposition 2.1 yield $\omega(x) > 0$ a.e. and $\omega(x)^{1-q'} \in A_{q'}^{\rho, \theta_4}$ so that by Lemma 2.2, for any nonnegative function $\|\varphi\|_{L_\omega^{q'}} \leq 1$, we then have

$$\int_{\mathbb{R}^n} |M_{V, \eta_1}(\varphi\omega)(x)|^{q'} \omega(x)^{1-q'} dx \leq C_q \int_{\mathbb{R}^n} |\varphi(x)|^{q'} \omega(x) dx = C_q,$$

where $\eta_1 = \bar{\eta}/(l_0 + 1)^3 > q\theta_4$ and hence

$$\begin{aligned} \int_{\mathbb{R}^n} |M_{V, \bar{\eta}} f(x)|_r^r \varphi(x) \omega(x) dx &\leq C \int_{\mathbb{R}^n} |f(x)|_r^r [M_{V, \eta_1}(\varphi\omega)(x)/\omega^{1/q}(x)] \omega^{1/q}(x) dx \\ &\leq C \left(\int_{\mathbb{R}^n} |f(x)|_r^{rq} \omega(x) dx \right)^{1/q}. \end{aligned} \quad (2.11)$$

In the first inequality of (2.11), we used the following fact that for any nonnegative measurable functions f, g , and $q > 1$, we have

$$\int_{\mathbb{R}^n} (M_{V, \bar{\eta}} f)^q g dx \leq C \int_{\mathbb{R}^n} f^q (M_{V, \eta_1} g) dx. \quad (2.12)$$

Taking the supremum in (2.11) over such φ then yields (2.2) for $1 < r \leq r_0$ upon taking $q = p/r$, and this together with the case $p = r$ provided in (2.3) yields (3.3) for $r_0 < r < p$ by application of Lemma 2.4. Thus, the proof of (a) and (b) is complete.

It remains to prove (2.12), let $\eta_2 = \eta_1(l_0 + 1) = \bar{\eta}/(l_0 + 1)^2$, we shall begin by proving

$$\int_{\mathbb{R}^n} (M_{V, \eta_2}^\Delta f)^q g dx \leq C \int_{\mathbb{R}^n} f^q (M_{V, \eta_1} g) dx. \quad (2.13)$$

We do this follows: Hold g fixed, and look at the mapping $T : f \rightarrow M_{V, \eta_2}^\Delta f$. Then (2.13) says that T is bounded from $L^q(\mathbb{R}^n, M_{V, \eta_1} g(x) dx)$ to $L^q(\mathbb{R}^n, g(x) dx)$. Clearly, T is bounded from $L^\infty(\mathbb{R}^n, M_{V, \eta_1} g(x) dx)$ to $L^\infty(\mathbb{R}^n, g(x) dx)$. If we can show that T is weak (1,1) type, then (2.13) holds by the Marcinkiewicz interpolation theorem.

Lemma 2.1 shows that $\{x \in \mathbb{R}^n : M_{V, \eta_2}^\Delta f(x) > \lambda\} = \bigcup_j Q_j$, where the Q_j are pairwise disjoint cubes satisfying the condition

$$\lambda \leq \frac{1}{\psi_{\eta_2}(Q_j)|Q_j|} \int_{Q_j} f(x) dx \leq 2^n (4n)^{\eta_2} \lambda.$$

Then

$$\begin{aligned} \int_{Q_j} g(y) dy &\leq \int_{Q_j} g(y) dy \frac{\lambda^{-1}}{\psi_{\eta_2}(Q_j)|Q_j|} \int_{Q_j} f(x) dx \\ &\leq C \lambda^{-1} \int_{Q_j} f(x) \left[\frac{1}{\Psi_{\eta_1}(Q_j)|Q_j|} \int_{Q_j} g(y) dy \right] dx \\ &\leq C \lambda^{-1} \int_{Q_j} f(x) M_{V, \eta_1} g(x) dx. \end{aligned}$$

Summing over j , we obtain

$$\int_{\{x \in \mathbb{R}^n : (M_{V, \eta_2}^\Delta f)(x) > \lambda\}} g(y) dy \leq C \int_{\mathbb{R}^n} f(x) M_{V, \eta_1} g(x) dx,$$

Thus, (2.13) holds. To complete the proof (2.12), we first define

$$M'_{V, \eta_3} f(x) = \sup_{r > 0} \frac{1}{(1 + r/\rho(x))^{\eta_3} |Q|} \int_{Q(x, r)} |f(y)| dy.$$

Obviously, $(4n)^{\bar{\eta}} C_0 M'_{V, \eta_3} f(x) \geq M_{V, \bar{\eta}} f(x)$, where $\eta_3 = \bar{\eta}/(l_0 + 1) = \eta_2(l_0 + 1)$.

Hence, to end the proof, it will suffice to show that

$$\{x \in \mathbb{R}^n : M'_{V, \eta_3} f(x) > c_0 \lambda\} \subset \bigcup_j 2Q_j, \quad (2.14)$$

where $c_0 = C_0^2 4^{l_0+1+n} (4n)^{\bar{\eta}}$.

Fix $x \in \bigcup_j 2Q_j$ and let Q be any cube centered at x . Let r denote the side length of Q , and choose $k \in \mathbb{Z}$ such that $2^{k-1} \leq r < 2^k$. Then Q intersects $m(\leq 2^n)$ dyadic cubes with sidelength 2^k ; call them $R_1 = R_1(x_1, 2^k), R_2 = R_2(x_2, 2^k), \dots, R_m = R_m(x_m, 2^k)$. Non of these cubes is contained in any of the Q'_j s, for otherwise we would have $x \in \bigcup_j (2Q_j)$. Hence

$$\begin{aligned} \frac{1}{(1 + r/\rho(x))^{\eta_3} |Q|} \int_{Q(x, r)} |f(y)| dy &= \frac{1}{(1 + r/\rho(x))^{\eta_3} |Q|} \sum_{i=1}^m \int_{Q \cap R_i} |f(y)| dy \\ &\leq \sum_{i=1}^m \frac{C_0 4^{l_0+1} 2^{kn}}{(1 + 2^k/\max_Q \rho(x))^{\eta_2} |Q| |R_i|} \int_{R_i} |f(y)| dy \\ &\leq 2^n 4^{l_0+1} C_0 m \lambda \leq 4^{l_0+1+n} C_0 \lambda. \end{aligned}$$

Thus, (2.14) holds, so (2.12) is proved. \square

3. Extrapolation theorems

In this section, \mathcal{F} will denote a family of order pairs of non-negative, measurable function (f, g) . If we say that for p , $0 < p < \infty$, and $\omega \in A_{\infty}^{p, \infty} = \bigcup_{p \geq 1} A_p^{p, \infty}$.

$$\int_{\mathbb{R}^n} f(x)^p \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p \omega(x), \quad (f, g) \in \mathcal{F},$$

we mean that this inequality holds for any $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and that the constant C depends only upon p and the $A_{\infty}^{p, \infty}$ constant of ω . We will make similar abbreviated statements involving Lorentz spaces. For vector-valued inequalities we will consider sequences $\{(f_j, g_j)\}$, where each pair (f_j, g_j) is contained in \mathcal{F} .

In addition, we will use following classes: given a pair of operators T and S , let $\mathcal{F}(T, S)$ denote the family of pairs of functions $(|Tf|, |Sf|)$, where f lies in the common domain of T and S , and the left-hand side of the corresponding inequality is finite. To achieve this, the function f may be restricted in some other way, e.g. $f \in C_0^\infty$. In this case we may indicate this by writing $\mathcal{F}(|Tf|, |Sf| : f \in C_0^\infty)$.

We can now state our main results in this paper.

Theorem 3.1. *Given a family \mathcal{F} , suppose that for some p_0 , $0 < p_0 < \infty$, and for every weight $\omega \in A_{\infty}^{\rho, \infty}$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0} \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} \omega(x), \quad (f, g) \in \mathcal{F}. \quad (3.1)$$

Then:

For all $0 < p < \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$

$$\int_{\mathbb{R}^n} f(x)^p \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p \omega(x) dx, \quad (f, g) \in \mathcal{F}. \quad (3.2)$$

For all $0 < p < \infty$, $0 < s \leq \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$

$$\|f\|_{L^{p,s}(\omega)} \leq C \|g\|_{L^{p,s}(\omega)}, \quad (f, g) \in \mathcal{F}. \quad (3.3)$$

For all $0 < p, q < \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$

$$\left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C \left\| \left(\sum_j (g_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}. \quad (3.4)$$

For all $0 < p, q < \infty$, $0 < s \leq \infty$, and $\omega \in A_{\infty}^{\rho, \infty}$

$$\left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p,s}(\omega)} \leq C \left\| \left(\sum_j (g_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p,s}(\omega)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}. \quad (3.5)$$

Our second main result shows that we can also extrapolate from an initial Lorentz space inequality.

Theorem 3.2. *Given a family \mathcal{F} , suppose that for some p_0 , $0 < p_0 < \infty$, and for every weight $\omega \in A_{\infty}^{\rho, \infty}$,*

$$\|f\|_{L^{p_0, \infty}(\omega)} \leq C \|g\|_{L^{p_0, \infty}(\omega)}, \quad (f, g) \in \mathcal{F}. \quad (3.6)$$

For all $0 < p < \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$

$$\|f\|_{L^{p, \infty}(\omega)} \leq C \|g\|_{L^{p, \infty}(\omega)}, \quad (f, g) \in \mathcal{F}. \quad (3.7)$$

Our third main result is a generalization of the A_p extrapolation theorem of Rubio de Francia.

Theorem 3.3. *Fix $\gamma \geq 1$ and r , $\gamma < r < \infty$. If T is a bounded operator on $L^r(\omega)$ for any $\omega \in A_{r/\gamma}^{\rho, \infty}$, with operator norm depending only the $A_{r/\gamma}$ constant of ω , then T is bounded on $L^p(\omega)$, $\gamma < p < \infty$, for any $\omega \in A_{p/\gamma}^{\rho, \infty}$.*

As a consequence of Theorem 3.3, we have the following result.

Corollary 3.1. *Fix $\gamma \geq 1$. Let $\gamma < p, q < \infty$ and T satisfy the conditions in Theorem 3.3. Then for any $\omega \in A_{p/\gamma}^{\rho, \infty}$ such that*

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}.$$

We shall adapt a similar argument in [6] for proving Theorems 3.1 and 3.2, and prove Theorem 3.3 by using an argument in [9]. We first give the proof of Theorem 3.1.

3.1. Proof of inequality (3.2)

We prove this inequality in two steps.

Step 1: We first show that hypothesis (3.1) is equivalent to the family of weighted inequalities with $A_1^{\rho, \infty}$ weights.

Proposition 3.1. *Hypothesis (3.1) of Theorem 3.1 is equivalent to the following: for all $0 < q < p_0$, $\omega \in A_1^{\rho, \infty}$, and $(f, g) \in \mathcal{F}$,*

$$\int_{\mathbb{R}^n} f(x)^q \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \omega(x) dx. \quad (3.8)$$

Proof of Proposition 3.1. We will prove that (3.1) implies (3.8). If (3.2) is proved, then the converse is proved. Fix $(f, g) \in \mathcal{F}$. Without loss of generality, we can assume that $g \in L^q(\omega)$ and $\|f\|_{L^q(\omega)} > 0$. Let $s = p_0/q$. Since $\omega \in A_1^{\rho, \infty}$, so there is a $\theta > 0$ such that $\omega \in A_1^{\rho, \theta} \subset A_{s'}^{\rho, \theta}$, and $M_{V, s\theta}$ is bounded on $L^{s'}(\omega)$ by Lemma 2.2, that is,

$$\|M_{V, s\theta} h\|_{L^{s'}(\omega)} \leq A \|h\|_{L^{s'}(\omega)},$$

for some $A > 0$. For $h \in L^{s'}(\omega)$, $h \geq 0$, we apply the algorithm of Rubio de Francia to define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M_{V, s\theta}^k h(x)}{(2A)^k},$$

where $M_{V, s\theta}^k$ is the operator $M_{V, s\theta}$ iterated k times if $k \geq 1$, and for $k = 0$ is just the identity. From the definition of \mathcal{R} , it easy to see that:

- (a) $h(x) \leq \mathcal{R}h(x)$.
- (b) $\|\mathcal{R}h\|_{L^{s'}(\omega)} \leq 2\|h\|_{L^{s'}(\omega)}$.
- (c) $M_{V, s\theta}(\mathcal{R}h)(x) \leq 2A\mathcal{R}h(x)$, so $\mathcal{R}h(x) \in A_1^{\rho, s\theta}$ with constant independent of h .

Since $f, g \in L^{s'}(\omega)$ and have positive norms, from (b), we then have

$$H(x) = \mathcal{R} \left(\left(\frac{f}{\|f\|_{L^{s'}(\omega)}} \right)^{\frac{q}{s'}} \left(\frac{g}{\|g\|_{L^{s'}(\omega)}} \right)^{\frac{q}{s'}} \right) (x) \in L^{s'}(\omega).$$

By (a),

$$\left(\frac{f}{\|f\|_{L^{s'}(\omega)}} \right)^{\frac{q}{s'}} \leq H(x), \quad \left(\frac{g}{\|g\|_{L^{s'}(\omega)}} \right)^{\frac{q}{s'}} \leq H(x), \quad (3.9)$$

So $H(x) > 0$ whenever $f(x) > 0$. Further, H is finite a.e. on the set where $\omega > 0$ because $h \in L^{s'}(\omega)$. Hence,

$$\int_{\mathbb{R}^n} f(x)^q \omega(x) dx \leq \left(\int_{\mathbb{R}^n} f(x)^{p_0} H(x)^{-s} \omega(x) dx \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^n} H(x)^{s'} \omega(x) dx \right)^{\frac{1}{s'}} := I \cdot II.$$

Obviously, $II \leq 4$ by (b).

To estimate I , since $\omega \in A_1^{\rho, \theta} \subset A_1^{\rho, s\theta}$, and $H \in A_1^{\rho, s\theta}$ by (c), so $wH^{-s} = wH^{1-(1+s)} \in A_{1+s}^{\rho, s\theta} \subset A_{\infty}^{\rho, \infty}$ by (v) of Proposition 2.1. on the other hand, by (3.9), we have

$$\int_{\mathbb{R}^n} f(x)^{p_0} H(x)^{-s} \omega(x) dx \leq \|f\|_{L^{s'}(\omega)}^{\frac{qs}{s'}} \int_{\mathbb{R}^n} f(x)^{p_0 - \frac{qs}{s'}} \omega(x) dx = \|f\|_{L^s(\omega)}^{qs} < \infty.$$

So, we can use (3.1); by (3.9), we get

$$I \leq \left(\int_{\mathbb{R}^n} g(x)^{p_0} H(x)^{-s} \omega(x) dx \right)^{\frac{1}{s}} \leq C \int_{\mathbb{R}^n} g(x)^p \omega(x) dx.$$

By I and II, we obtain the desired result.

Step 2: We now show that for all $0 < p < \infty$ and for every $\omega \in A_{\infty}^{\rho, \infty}$, (3.2) holds. Fix $0 < p < \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$. Assume that $(f, g) \in \mathcal{F}$ with $f \in L^p(\omega)$ and $g \in L^p(\omega)$. By (i) of Proposition 2.1, we know that $A_{p_1}^{\rho, \theta} \subset A_{p_2}^{\rho, \theta}$ if $1 \leq p_1 \leq p_2$, there exist $\theta > 0$ and $0 < q < \min\{p, p_0\}$ such that $\omega \in A_{p/q}^{\rho, \theta}$. Let $r = p/q > 1$. Since $\omega \in A_r^{\rho, \theta}$, then $\omega^{1-r'} \in A_{r'}^{\rho, \theta}$ by (ii) of Proposition 2.1. Given $h \in L^{r'}(\omega^{1-r'})$, $h \geq 0$, we use the algorithm of Rubio de Francia to define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M_{V, r\theta}^k h(x)}{(2B)^k},$$

where B is the operator norm of $M_{V, r\theta}$ on $L^{r'}(\omega^{1-r'})$; this is finite since $\omega^{1-r'} \in A_{r'}^{\rho, \theta}$. Then

$$(a) \quad h(x) \leq \mathcal{R}h(x).$$

$$(b) \quad \|\mathcal{R}h\|_{L^{r'}(\omega^{1-r'})} \leq 2\|h\|_{L^{r'}(\omega^{1-r'})}.$$

$$(c) \quad M_{V, sr}(\mathcal{R}h)(x) \leq 2B\mathcal{R}h(x), \text{ so } \mathcal{R}h(x) \in A_1^{\rho, r\theta} \text{ with constant independent of } h.$$

By duality

$$\|f\|_{L^p(\omega)}^q = \|f^q\|_{L^r(\omega)} = \sup_{\|h\|_{L^{r'}(\omega)} \leq 1} \int_{\mathbb{R}^n} f(x)^q h(x) \omega(x) dx.$$

Fix such a function $h \geq 0$. Then $h\omega \in L^{r'}(\omega^{1-r'})$ and $\|h\omega\|_{L^{r'}(\omega^{1-r'})} = \|h\|_{L^{r'}(\omega)} = 1$. By (c), $\mathcal{R}(h\omega) \in A_1^{\rho, r\theta}$. By (a) and Proposition 3.1, we then have

$$\int_{\mathbb{R}^n} f(x)^q h(x) \omega(x) dx \leq \int_{\mathbb{R}^n} f(x)^q \mathcal{R}(h\omega)(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}(h\omega)(x) dx,$$

provided that the middle term is finite, this is obvious.

The same argument also holds for g instead of f . Hence,

$$\int_{\mathbb{R}^n} f(x)^q h(x) \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}(h\omega)(x) dx \leq C \|g\|_{L^p(\omega)}^q.$$

From this, we obtain the desired result. \square

3.2. Proof of inequality (3.3)

We need two lemmas. We first give a result about the operator M_ω defined by

$$M_\omega(f)(x) = \sup_{x \in B} \frac{1}{\omega(5B)} \int_B |f(x)| \omega(x) dx.$$

Lemma 3.1 ([23]). *Let $1 \leq p < \infty$. If $\omega \in A_\infty^{\rho, \infty}$, then*

$$\omega(\{x \in \mathbb{R}^n : M_\omega f(x) > \lambda\}) \leq C \left(\frac{\|f\|_{L^p(\omega)}}{\lambda} \right)^p, \quad \forall \lambda > 0, \quad \forall f \in L^p(\omega).$$

In particular, for $1 < p \leq fz$,

$$\|M_\omega f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

Given two weights u and v , we say that $u \in A_1(v)$ if for every x , $M_v u(x) \leq C u(x)$.

Lemma 3.2. *If $\omega_1 \in A_p^{\rho, \theta}$, $1 \leq p \leq \infty$, and $\omega_2 \in A_1(\omega_1)$, then $\omega_1 \omega_2 \in A_p^{\rho, \theta p}$.*

Proof. If $\omega_2 \in A_1(\omega_1)$, then for any ball B

$$\begin{aligned} \frac{1}{(\Psi_\theta(B))^{p^2} |B|} \int_B \omega_1(x) \omega_2(x) dx &= \frac{\omega_1(5B)}{(\Psi_\theta(B))^{p^2} |B|} \frac{1}{\omega_1(5B)} \int_B \omega_2(x) \omega_1(x) dx \\ &\leq C \frac{\omega_1(5B)}{(\Psi_\theta(B))^{p^2} |B|} \text{ess inf}_B \omega_2 \\ &\leq C \frac{\omega_1(B)}{(\Psi_\theta(B))^p |B|} \text{ess inf}_B \omega_2, \end{aligned}$$

in the last inequality, we used the following fact (see [23])

$$\omega_1(5B) \leq C(\Psi_\theta(B))^p \omega_1(B).$$

On the other hand,

$$\left(\frac{1}{|B|} \int_B (\omega_1(x) \omega_2(x))^{-\frac{1}{p-1}} dx \right)^{p-1} \leq \left(\frac{1}{|B|} \int_B \omega_1(x)^{-\frac{1}{p-1}} dx \right)^{p-1} (\text{ess inf}_B \omega_2)^{-1}.$$

From two inequalities above, we get the desired result. \square

Proof of (3.3). Fix $p, s, \omega \in A_{\infty}^{\rho, \infty}$ and $(f, g) \in \mathcal{F}$ with $f, g \in L^{p, s}(\omega)$. Fix $0 < q < \min\{p, s\}$ and set $r = p/q > 1$, $\tilde{r} = s/q > 1$. (If $s = \infty$, take $0 < q < p$ and $\tilde{r} = \infty$.) Then

$$\|f\|_{L^{p, s}(\omega)}^q = \|f^q\|_{L^{r, \tilde{r}}(\omega)} = \sup_h \int_{\mathbb{R}^n} f(x)^q h(x) \omega(x) dx,$$

where the supremum is taken over all $h \in L^{r', \tilde{r}}(\omega)$ with $h \geq 0$ and $\|h\|_{L^{r', \tilde{r}}(\omega)} = 1$. Fix such a function h . Using the algorithm of Rubio de Francia to define

$$\mathcal{R}_{\omega} h(x) = \sum_{k=0}^{\infty} \frac{M_{\omega}^k h(x)}{(2A_{\omega})^k},$$

where A_{ω} is the operator norm of M_{ω} on $L^{r', \tilde{r}}(\omega)$ endowed with norm equivalent to $\|\cdot\|_{L^{r', \tilde{r}}(\omega)}$. Since M_{ω} is bounded on $L^p(\omega)$ by Lemma 3.1, and by Marcinkiewicz interpolation in the scale of Lorentz space, it is bounded on $L^{r', \tilde{r}}(\omega)$. Then,

- (a) $h(x) \leq \mathcal{R}_{\omega} h(x)$.
- (b) $\|\mathcal{R}_{\omega} h\|_{L^{r', \tilde{r}}(\omega^{1-r'})} \leq C \|h\|_{L^{r', \tilde{r}}(\omega^{1-r'})} = C$.
- (c) $M_{V, s\theta}(\mathcal{R}h)(x) \leq 2A_{\omega} \mathcal{R}h(x)$, so $\mathcal{R}_{\omega} h(x) \in A_1(\omega)$ with constant independent of h .

By Lemma 3.2, $\omega \mathcal{R}_{\omega} h \in A_{\infty}^{\rho, \infty}$. As above, (3.2) holds with exponent q and the $A_{\infty}^{\rho, \infty}$ weight $\omega \mathcal{R}_{\omega} h$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)^q h(x) \omega(x) dx &\leq \int_{\mathbb{R}^n} f(x)^q \mathcal{R}_{\omega} h(x) \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}_{\omega} h(x) \omega(x) dx \\ &\leq C \|g^q\|_{L^{r, \tilde{r}}(\omega)} \|\mathcal{R}_{\omega} h\|_{L^{r', \tilde{r}}(\omega)} \leq C \|g\|_{L^{r, \tilde{r}}(\omega)}^q, \end{aligned}$$

since

$$\int_{\mathbb{R}^n} f(x)^q \mathcal{R}_{\omega} h(x) \omega(x) dx \leq \|f^q\|_{L^{r, \tilde{r}}(\omega)} \|\mathcal{R}_{\omega} h\|_{L^{r', \tilde{r}}(\omega)} \leq C \|f\|_{L^{r, \tilde{r}}(\omega)}^q < \infty.$$

Thus, the desired inequality is obtained. \square

3.3. Proof of inequalities (3.4) and (3.5)

Fix $0 < q < \infty$. It suffices to prove the vector-valued inequalities only for finite sums by the monotone convergence theorem. Fix $N \geq 1$ and define

$$f_q(x) = \left(\sum_{j=1}^N f_j(x)^q \right)^{\frac{1}{q}}, \quad g_q(x) = \left(\sum_{j=1}^N g_j(x)^q \right)^{\frac{1}{q}},$$

where $\{(f_j, g_j)\}_{j=1}^N \subset \mathcal{F}$. Now form a new family \mathcal{F}_q consisting of the pairs (f_q, g_q) . Then, for every $\omega \in A_\infty^{\rho, \infty}$ and $(f_q, g_q) \in \mathcal{F}_q$, by (3.2) we get

$$\|f_q\|_{L^q(\omega)}^q = \sum_{j=1}^N \int_{\mathbb{R}^n} f_j(x)^q \omega(x) dx \leq C \sum_{j=1}^N \int_{\mathbb{R}^n} g_j(x)^q \omega(x) dx = C \|g_q\|_{L^q(\omega)}^q,$$

which implies that the hypotheses of Theorem 3.1 are fulfilled by \mathcal{F}_q with $p_0 = q$. Hence, by (3.2) and (3.3), for all, $0 < p < \infty$, $0 < s \leq \infty$, $\omega \in A_\infty^{\rho, \infty}$, and $(f_q, g_q) \in \mathcal{F}_q$, $\|f_q\|_{L^p(\omega)} \leq C \|g_q\|_{L^p(\omega)}$ and $\|f_q\|_{L^{p,s}(\omega)} \leq C \|g_q\|_{L^{p,s}(\omega)}$. \square

3.4. Proof of Theorem 3.2

Similar to the proof of Theorem 3.1, and adapting the same argument of Theorem 2.2 in [6], we omit the details here.

3.4. Proof of Theorem 3.3

We first need the following lemma, which is different from Lemma 2.2.

Lemma 3.3([23]). *Let $1 \leq p < \infty$ and suppose that $\omega \in A_p^{\rho, \theta}$. If $p < p_1 < \infty$, then*

$$\int_{\mathbb{R}^n} |M_{V, \theta} f(x)|^{p_1} \omega(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_1} \omega(x) dx.$$

Proof. We only consider the case $\gamma = 1$, another case $\gamma > 1$ is similar. We first show that if $1 < q < r$ and $\omega \in A_1^{\rho, \infty}$ then T is bounded on $L^q(\omega)$. Without loss of generality, we assume $\omega \in A_1^{\rho, \eta}$ for some $\eta > 0$. By (vi) of Proposition 2.1 the function $M_{V, \eta}^{(r-q)/(r-1)}$ is in $A_1^{\rho, \eta}$, and $\omega(M_{V, \eta} f)^{q-r} \in A_r^{\rho, \eta}$ by (v) of Proposition 2.1. Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf|^q \omega &= \int_{\mathbb{R}^n} |Tf|^q (M_{V, \eta} f)^{-(q-r)q/r} (M_{V, \eta} f)^{(q-r)q/r} \omega \\ &\leq \left(\int_{\mathbb{R}^n} |Tf|^r \omega (M_{V, \eta} f)^{q-r} \right)^{q/r} \left(\int_{\mathbb{R}^n} (M_{V, \eta} f)^q \omega \right)^{(r-q)/r} \\ &\leq \left(\int_{\mathbb{R}^n} |f|^r \omega (M_{V, \eta} f)^{q-r} \right)^{q/r} \left(\int_{\mathbb{R}^n} |f|^q \omega \right)^{(r-q)/r} \\ &\leq C \int_{\mathbb{R}^n} |f|^q \omega, \end{aligned}$$

the second inequality holds by our hypothesis on T and by Lemma 3.3 (since $\omega \in A_1^{\rho, \eta}$), and the third inequality holds since $|f(x)| \leq M_{V, \eta} f(x)$ a.e. for any $\eta \geq 0$, so $M_{V, \eta} f(x)^{q-r} \leq |f(x)|^{q-r}$ a.e.

Given any $1 < p < \infty$ and $\omega \in A_p^{\rho, \theta}$, by (iii) of Proposition 2.1 there exists $q > 1$ and $\theta_1 \geq \theta$ such that $\omega \in A_{p/q}^{\rho, \theta_1}$, hence we only need to prove that T is bounded on $L^p(\omega)$ if $\omega \in A_{p/q}^{\rho, \theta_1}$.

Fix $\omega \in A_{p/q}^{\rho, \theta_1}$. Then by duality there exists $u \in L^{(p/q)'}(\omega)$ with norm 1 such that

$$\left(\int_{\mathbb{R}^n} |Tf|^p \omega \right)^{q/p} = \int_{\mathbb{R}^n} |Tf|^q \omega u.$$

For any $s > 1$, $\omega u \leq M_{V,\eta}((\omega u)^s)^{1/s}$ for any $\eta > 0$ and $M_{V,\eta}((\omega u)^s)^{1/s} \in A_1^{\rho, \eta}$. Hence, by the first part of the proof,

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf|^q \omega u &\leq \int_{\mathbb{R}^n} |Tf|^q M_{V,\eta}((\omega u)^s)^{1/s} \\ &\leq C \int_{\mathbb{R}^n} |f|^q M_{V,\eta}((\omega u)^s)^{1/s} \\ &= C \int_{\mathbb{R}^n} |f|^q \omega^{q/p} M_{V,\eta}((\omega u)^s)^{1/s} \omega^{-q/p} \\ &\leq C \left(\int_{\mathbb{R}^n} |f|^p \omega \right)^{q/p} \left(\int_{\mathbb{R}^n} M_{V,\eta}((\omega u)^s)^{(p/q)'/s} \omega^{1-(p/q)'} \right)^{1/(p/q)'} \end{aligned}$$

Since $\omega \in A_{p/q}^{\rho, \theta_1}$, then $\omega^{1-(p/q)'} \in A_{(p/q)'}^{\rho, \theta_1}$ by (ii) of Proposition 2.1. Therefore, if take s sufficient close to 1, then there exists θ_s such that $\omega^{1-(p/q)'} \in A_{(p/q)'/s}^{\rho, \theta_s}$ by (iii) of Proposition 2.1. If choosing $\eta = ((p/q)'/s)\theta_s$, then by Lemma 2.2 the second integral is dominated by

$$C \int_{\mathbb{R}^n} (\omega u)^{(p/q)'} \omega^{1-(p/q)'} = C.$$

The proof is complete. \square

4. Some applications

4.1. Schrödinger type operators

Let T be a Schrödinger type operators. From Theorem 3.1 in [23] we know that for all $0 < p < \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$, for any $\eta > 0$, then there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_{\infty}^{\rho, \infty}$ constant of ω such that

$$\|Tf\|_{L^p(\omega)} \leq C \|M_{V,\eta}f\|_{L^p(\omega)}.$$

By applying Theorem 3.1 to the family $\mathcal{F}_{\eta}(|Tf|, M_{V,\eta}f : f \in C_0^{\infty})$, we obtain that

For all $0 < p, q < \infty$ and $\omega \in A_{\infty}^{\rho, \infty}$

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C \left\| \left(\sum_j (M_{V,\eta}f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}_{\eta}. \quad (4.1)$$

For all $0 < p, q < \infty$, $0 < s \leq \infty$, and $\omega \in A_{\infty}^{\rho, \infty}$

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p,s}(\omega)} \leq C \left\| \left(\sum_j (M_{V,\eta}f_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p,s}(\omega)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}_{\eta}. \quad (4.2)$$

If we combine them with Theorem 2.1, we have the following inequalities:

If $1 < q < \infty$, then for every $\omega \in A_1^{\rho, \infty}$, there exists a constant C depending only on η, q, C_0, l_0 and the $A_1^{\rho, \infty}$ constant of ω such that

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^{1, \infty}(\omega)} \leq C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^1(\omega)}, \quad (4.3)$$

If $1 < q < \infty$, and $1 < p < \infty$, then for every $\omega \in A_p^{\rho, \infty}$, there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_p^{\rho, \infty}$ constant of ω such that

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C \left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}. \quad (4.4)$$

Let T be a Schrödinger type operators as above. From Theorem 3.1 in [23] we have that for all $0 < p < \infty$ and $\omega \in A_\infty$, for any $\eta > 0$, then there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_\infty^{\rho, \infty}$ constant of ω such that

$$\|[b, T]f\|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \|M_{V, \eta}(M_{V, \eta}f)\|_{L^p(\omega)}.$$

By applying Theorem 3.1 to the family $\mathcal{F}_\eta(|[b, T]f|, M_{V, \eta}f : f \in C_0^\infty)$, we obtain that

For all $0 < p, q < \infty$ and $\omega \in A_\infty^{\rho, \infty}$

$$\left\| \left(\sum_j |[b, T]f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \left\| \left(\sum_j (M_{V, \eta}f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}_\eta. \quad (4.5)$$

For all $0 < p, q < \infty$, $0 < s \leq \infty$, and $\omega \in A_\infty^{\rho, \infty}$

$$\left\| \left(\sum_j |[b, T]f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p, s}(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \left\| \left(\sum_j (M_{V, \eta}f_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p, s}(\omega)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}_\eta, \quad (4.6)$$

where the new space $BMO_\theta(\rho)$ introduced in [4] as follows

$$\|f\|_{BMO_\theta(\rho)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(x) - f_B| dx < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(y) dy$ and $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, $B = B(x_0, r)$ and $\theta > 0$. Let $BMO_\infty(\rho)$ denote $\bigcup_{\theta > 0} BMO_\theta(\rho)$

If we combine them with Theorem 2.1, we have the following inequality: If $1 < q < \infty$, and $1 < p < \infty$, then for every $\omega \in A_p^{\rho, \infty}$, there exists a constant C depending only on η, p, q, C_0, l_0 and the $A_p^{\rho, \infty}$ constant of ω such that

$$\left\| \left(\sum_j |[b, T]f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(\omega)}. \quad (4.7)$$

We remark that these inequalities (4.1)-(4.7) are all new.

Next we consider another class $V \in B_q$ for $n/2 \leq q$ for Riesz transforms associated to Schrödinger operators. Let $T_1 = (-\Delta + V)^{-1}V$, $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$ and $T_3 = (-\Delta + V)^{-1/2}\nabla$. By using Theorem 3.3 in [24] and Corollary 3.3, we have

Theorem 4.1. *Suppose $V \in B_q$ and $q \geq n/2$. Then*

(i) *If $q' < p, r < \infty$ and $\omega \in A_{p/q'}^{\rho, \infty}$,*

$$\| |T_1 f|_r \|_{L^p(\omega)} \leq C \| |f|_r \|_{L^p(\omega)};$$

(ii) *If $(2q)' < p, r < \infty$ and $\omega \in A_{p/(2q)'}^{\rho, \infty}$,*

$$\| |T_2 f|_r \|_{L^p(\omega)} \leq C \| |f|_r \|_{L^p(\omega)};$$

(iii) *If $p'_0 < p, r < \infty$ and $\omega \in A_{p/p'_0}^{\rho, \infty}$, where $1/p_0 = 1/q - 1/n$ and $n/2 \leq q < n$,*

$$\| |T_3 f|_r \|_{L^p(\omega)} \leq C \| |f|_r \|_{L^p(\omega)}.$$

Let $T_1^* = V(-\Delta + V)^{-1}$, $T_2^* = V^{1/2}(-\Delta + V)^{-1/2}$ and $T_3^* = \nabla(-\Delta + V)^{-1/2}$. By duality we can easily get the following results.

Corollary 4.1. *Suppose $V \in B_q$ and $q \geq n/2$. Then*

(i) *If $1 < p, r < q$ and $\omega^{-\frac{1}{p-1}} \in A_{p'/q'}^{\rho, \infty}$,*

$$\| |T_1^* f|_r \|_{L^p(\omega)} \leq C \| |f|_r \|_{L^p(\omega)};$$

(ii) *If $1 < p, r < 2q$ and $\omega^{-\frac{1}{p-1}} \in A_{p'/(2q)'}^{\rho, \infty}$,*

$$\| |T_2^* f|_r \|_{L^p(\omega)} \leq C \| |f|_r \|_{L^p(\omega)};$$

(iii) *If $1 < p, r < p_0$ and $\omega^{-\frac{1}{p-1}} \in A_{p'/p'_0}^{\rho, \infty}$, where $1/p_0 = 1/q - 1/n$ and $n/2 \leq q < n$,*

$$\| |T_3^* f|_r \|_{L^p(\omega)} \leq C \| |f|_r \|_{L^p(\omega)}.$$

Let T_1 , T_2 and T_3 be above. By using Theorem 4.5 in [24] and Corollary 3.3, we have

Theorem 4.2. *Suppose $V \in B_q$ and $q \geq n/2$. Let $b \in BMO_\infty(\rho)$. Then*

(i) *If $q' < p, r < \infty$ and $\omega \in A_{p/q'}^{\rho, \infty}$,*

$$\| | [b, T_1] f |_r \|_{L^p(\omega)} \leq C \| b \|_{BMO_\infty(\rho)} \| |f|_r \|_{L^p(\omega)};$$

(ii) *If $(2q)' < p, r < \infty$ and $\omega \in A_{p/(2q)'}^{\rho, \infty}$,*

$$\| | [b, T_2] f |_r \|_{L^p(\omega)} \leq C \| b \|_{BMO_\infty(\rho)} \| |f|_r \|_{L^p(\omega)};$$

(iii) If $p'_0 < p, r < \infty$ and $\omega \in A_{p/p'_0}^{\rho, \infty}$, where $1/p_0 = 1/q - 1/n$ and $n/2 \leq q < n$,

$$\| [b, T_3]f |_r \|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \|f |_r \|_{L^p(\omega)}.$$

Let T_1^* , T_2^* and T_3^* be above. By duality we can easily get the following results.

Corollary 4.2. Suppose $V \in B_q$ and $q \geq n/2$. Let $b \in BMO_\infty(\rho)$. Then

(i) If $1 < p, r < q$ and $\omega^{-\frac{1}{p-1}} \in A_{p'/q'}^{\rho, \infty}$,

$$\| [b, T_1^*]f |_r \|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \|f |_r \|_{L^p(\omega)};$$

(ii) If $1 < p, r < 2q$ and $\omega^{-\frac{1}{p-1}} \in A_{p'/(2q)'}^{\rho, \infty}$,

$$\| [b, T_2^*]f |_r \|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \|f |_r \|_{L^p(\omega)};$$

(iii) If $1 < p, r < p_0$ and $\omega^{-\frac{1}{p-1}} \in A_{p'/p'_0}^{\rho, \infty}$, where $1/p_0 = 1/q - 1/n$ and $n/2 \leq q < n$,

$$\| [b, T_3^*]f |_r \|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \|f |_r \|_{L^p(\omega)}.$$

Finally, we consider the Littlewood-Paley g function related to Schrödinger operators is defined by

$$g(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-tL}(f)(x) \right|^2 t dt \right)^{1/2},$$

and the commutator g_b of g with $b \in BMO(\rho)$ is defined by

$$g_b(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-tL}((b(x) - b(\cdot))f)(x) \right|^2 t dt \right)^{1/2}.$$

The maximal operator of the diffusion semi-group is defined by

$$T^*f(x) = \sup_{t>0} |e^{-tL}f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^n} k_t(x, y) f(y) dy \right|,$$

and it's commutator

$$T_b^*f(x) = \sup_{t>0} \left| \int_{\mathbb{R}^n} k_t(x, y) (b(x) - b(y)) f(y) dy \right|,$$

where k_t is the kernel of the operator e^{-tL} , $t > 0$.

By Combining Theorems 1 and 2 in [5] and Theorems 1.1 and 3.1 in [24] and Corollary 3.3 together, we have

Theorem 4.3. Let $b \in BMO_\infty(\rho)$ and T, T_b^* , g and g_b be as above.

(i) If $1 < p, r < \infty$, $\omega \in A_p^{\rho, \infty}$, then there exists a constant C such that

$$\| |g(f)|_r \|_{L^p(\omega)} + \| |T^*f |_r \|_{L^p(\omega)} \leq C \|f |_r \|_{L^p(\omega)}.$$

(ii) If $1 < p, r < \infty$, $\omega \in A_p^{\rho, \infty}$, then there exists a constant C such that

$$\| |g_b(f)|_r \|_{L^p(\omega)} + \| |T_b^*f |_r \|_{L^p(\omega)} \leq C \|b\|_{BMO_\infty(\rho)} \|f |_r \|_{L^p(\omega)}.$$

4.2. Pseudo-differential operators

Let m be real number. Following [22], a symbol in $S_{1,\delta}^m$ is a smooth function $\sigma(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β the following estimate holds:

$$|D_x^\alpha D_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|+\delta|\alpha|},$$

where $C_{\alpha,\beta} > 0$ is independent of x and ξ . A symbol in $S_{1,\delta}^{-\infty}$ is one which satisfies the above estimates for each real number m .

The operator T given by

$$Tf(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi$$

is called a pseudo-differential operator with symbol $\sigma(x, \xi) \in S_{1,\delta}^m$, where f is a Schwartz function and \hat{f} denotes the Fourier transform of f . As usual, $L_{1,\delta}^m$ will denote the class of pseudo-differential operators with symbols in $S_{1,\delta}^m$.

We in [25] studied weighted inequalities for a class of pseudo-differential operators with symbols in $S_{1,\delta}^0$ with $0 < \delta < 1$. More precisely, we have the following result.

Lemma 4.1. *Let T be a pseudo-differential operators with symbols in $S_{1,\delta}^0$ with $0 < \delta < 1$, and let $0 < \delta < 1$, for any $\eta > 0$. Then there exists a constant $C > 0$ such that*

$$M_{\varphi,\delta,\eta}^\sharp(Tf)(x) \leq CM_{\varphi,\eta}(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n$$

for any smooth function f with compact support, where $\varphi_\eta(Q) = (1+r)^\eta$ with $r = |Q|^{1/n}$, and

$$M_{\varphi,\eta}f(x) = \sup_{x \in Q} \frac{1}{\varphi_\eta(Q)|Q|} \int_Q |f(y)| dy,$$

and $M_{\varphi,\delta,\eta}^\sharp f(x) = M_{\varphi,\eta}^\sharp(|f|^\delta)^{1/\delta}(x)$, and the sharp maximal operator $M_{\varphi,\eta}^\sharp f(x)$ is defined by

$$\begin{aligned} M_{\varphi,\eta}^\sharp f(x) &:= \sup_{x \in Q, r < 1} \frac{1}{|Q|} \int_{Q_{x_0}} |f(y) - f_Q| dy + \sup_{x \in Q, r \geq 1} \frac{1}{\varphi_\eta(Q)|Q|} \int_{Q_{x_0}} |f| dy \\ &\simeq \sup_{x \in Q, r < 1} \inf_C \frac{1}{|Q|} \int_{Q_{x_0}} |f(y) - C| dy + \sup_{x \in Q, r \geq 1} \frac{1}{\varphi_\eta(Q)|Q|} \int_{Q_{x_0}} |f| dy \end{aligned}$$

where Q_{x_0} denotes cubes $Q(x_0, r)$ and $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$.

Lemma 4.2([25]). *Let $0 < p$, $\delta < \infty$ and $\omega \in A_\infty^{1,\infty} = A_\infty^{\rho,\infty}$ with $\rho = 1$. There exists a positive constant C such that*

$$\int_{\mathbb{R}^n} M_{\varphi,\delta,\eta}f(x)^p \omega(x) dx \leq C \int_{\mathbb{R}^n} M_{\varphi,\delta,\eta}^\sharp f(x)^p \omega(x) dx,$$

where $M_{\varphi,\delta,\eta}f(x) = M_{\varphi,\eta}(|f|^\delta)^{1/\delta}(x)$,

From Lemmas 4.1 and 4.2, we have that for all $0 < p < \infty$ and $\omega \in A_\infty^{1,\infty}$, for any $\eta > 0$, then there exists a constant C depending only on η, p and the $A_\infty^{1,\infty}$ constant of ω such that

$$\|Tf\|_{L^p(\omega)} \leq C \|M_{\varphi,\eta} f\|_{L^p(\omega)}.$$

From this, we can get the vector-valued estimates (3.4) and (3.5) which are new.

Next, we consider the multilinear pseudo-differential operators, that is, T is an m -linear operator such that T are initially defined on the m -fold product of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and take their values into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. We will assume that the distributional kernel on $(\mathbb{R}^n)^{m+1}$ of the operator coincides away from the diagonal $y_0 = y_1 = y_2 = \dots = y_m$ in $(\mathbb{R}^n)^{m+1}$ with a function K for integer $m \geq 1$ so that

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

whenever f_1, \dots, f_m are C^∞ functions with compact support and $x \notin \bigcap_{j=1}^m \text{supp} f_j$. Moreover, we will assume that the function K satisfies the following estimates for any $N \geq 0$

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{C_N}{(1 + \sum_{k,l=0}^m |y_k - y_l|)^N (\sum_{k,l=0}^m |y_k - y_l|)^{mn}}, \quad (4.8)$$

and, for some $\epsilon > 0$ and any $N \geq 0$,

$$\begin{aligned} |K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_l)| \\ \leq \frac{C_N |y_j - y'_j|^\epsilon}{(1 + \sum_{k,l=0}^m |y_k - y_l|)^N (\sum_{k,l=0}^m |y_k - y_l|)^{mn+\epsilon}}, \end{aligned} \quad (4.9)$$

provided that $0 \leq j \leq m$ and $|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_k - y'_k|$. When $N = 0$ in (4.8) and (4.9), such kernels are called m -linear Calderón-Zygmund kernels and the collections is denoted in [11] by $m\text{-}CZK$. For these operators above, a boundedness estimate

$$T : L^{q_1} \times \dots \times L^{q_m} \rightarrow L^q,$$

for $1 < q_1, \dots, q_m < \infty$ and

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{1}{q}, \quad (4.10)$$

implies the boundedness of the operator for all possible exponents in such range of values. Moreover, it will be important for purpose the following end-point estimate also satisfied by such operators:

$$L^{q_1} \times \dots \times L^{q_m} \rightarrow L^{q,\infty},$$

for $1 \leq q_1, \dots, q_m < \infty$ satisfying (4.10). In particular, it will be relevant the case

$$L^1 \times \dots \times L^1 \rightarrow L^{1/m,\infty},$$

which extends the classical result in the linear case $T : L^1 \rightarrow L^{1,\infty}$; see [11].

A typical example, Let T is a bilinear pseudo-differential operator with symbols belonging to $SB_{1,0}^0$; see [3]. From [27], we know that the kernel of T satisfies (4.8) and (4.9), and it is bounded from $L^1 \rightarrow L^{1/2} \times L^{1/2}$; see [3].

We next give a estimate for a multilinear pseudo-differential operator.

Lemma 4.3. *Let T be a multilinear pseudo-differential operator as above. Let $0 < \delta < 1/m$, $\eta_j > 0$ for $j = 1, \dots, m$, and $\eta = \sum_{j=1}^m \eta_j$. Then there exists a constant $C > 0$ such that*

$$M_{\delta,\eta}^\sharp(T\vec{f})(x) \leq C \prod_{j=1}^m M_{\varphi,\eta_j}(f_j)(x), \quad \text{a.e. } x \in \mathbb{R}^n \quad (4.11)$$

for any smooth vector function $\vec{f} = (f_1, f_2, \dots, f_m)$ with compact support.

Proof. Let \vec{f} be any smooth vector function. Let $x \in Q = Q(x_0, r)$. Write each $\vec{f} = \vec{f}^0 + \vec{f}^\infty$, where $\vec{f}^0 = \vec{f}\chi_{2Q} = (f_1\chi_{2Q}, \dots, f_m\chi_{2Q})$. Set

$$C_Q = (T(\vec{f}^\infty))_Q = T(f_1^\infty, \dots, f_m^\infty)_Q.$$

It is easy to see that

$$|T(\vec{f})(z) - C_Q| \leq |T(\vec{f}^\infty)(z) - C_Q| + C_{qm} \sum_m T(f_1^{r_1}, \dots, f_m^{r_m})(z), \quad (4.12)$$

where in the last sum each $r_j = 0$ or ∞ and in each term there is at least are $r_j = 0$.

To prove (4.11), we consider two cases about r , that is, $r < 1$ and $r \geq 1$.

Case 1. when $r < 1$. Using the regularity of the kernel (4.9) and $0 < \delta < 1/m < 1$, by Minkowski's inequality, we get

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q |T^\delta(\vec{f}^\infty)(z) - C_Q^\delta| dz \right)^{1/\delta} \\ & \leq \left(\frac{1}{|Q|} \int_Q |T(\vec{f}^\infty)(z) - (T(\vec{f}^\infty))_Q|^\delta dz \right)^{1/\delta} \\ & = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q T(\vec{f}^\infty)(z) - T(\vec{f}^\infty)(y) dy \right| dz \\ & \leq \frac{C}{|Q|} \int_Q \frac{1}{|Q|} \int_Q \int_{(\mathbb{R}^n)^m \setminus (2Q)^m} \\ & \quad \times (K(z, \vec{w}) - k(y, \vec{w})) \prod_{j=1}^m f_j(w_j) d\vec{w} dy dz \\ & \leq \frac{C_N}{|Q|} \int_Q \frac{1}{|Q|} \int_Q \int_{(\mathbb{R}^n)^m \setminus (2Q)^m} \frac{1}{(1 + |y - w_1| + \dots + |y - w_m|)^N} \\ & \quad \times \frac{|y - z|^\epsilon}{(|y - w_1| + \dots + |y - w_m|)^{mn+\epsilon}} \prod_{j=1}^m |f_j(w_j)| d\vec{w} dy dz \end{aligned}$$

$$\begin{aligned}
 &\leq C_N |Q|^{\frac{\epsilon}{n}} \prod_{j=1}^m \left(\int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(w_j)|}{(1 + |x - w_j|)^{\eta_j} |x - w_j|^{n + \frac{\epsilon}{m}}} d\vec{w} \right) \\
 &\leq C_N \prod_{j=1}^m |Q|^{\frac{\epsilon}{mn}} \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(w_j)|}{(1 + |x - w_j|)^{\eta_j} |x - w_j|^{n + \frac{\epsilon}{m}}} dw_j \\
 &\leq C_N \prod_{j=1}^m M_{\varphi, \eta_j}(f_j)(x),
 \end{aligned}$$

if taking $N = m\eta$.

The above computations gives the correct restimates for the first term in the right hand side of (4.2). To estimate the sum in the right hand side of (4.12) we distinguish between two kinds of term . One, in which at least one of the $k_j = \infty$, and one final term in which all the $k_j = 0$. A typical representative of the first kind of term is $T(f_1^\infty, \dots, f_i^\infty, f_{i+1}^0, \dots, f_m^0)(z)$. Using the notation $R_i = (\mathbb{R}^n \setminus 2Q)^i \times (2Q)^{m-i}$, by Minkowski's inequality, we have

$$\begin{aligned}
 &\left(\frac{1}{|Q|} \int_Q |T(f_1^\infty, \dots, f_i^\infty, f_{i+1}^0, \dots, f_m^0)(z)|^\delta dz \right)^{1/\delta} \\
 &\leq \frac{1}{|Q|} \int_Q |T(f_1^\infty, \dots, f_i^\infty, f_{i+1}^0, \dots, f_m^0)(z)| dz \\
 &\leq \frac{1}{|Q|} \int_Q \left| \int_{(\mathbb{R}^n)^m} k(x, \vec{y}) f_1^\infty(y_1), \dots, f_i^\infty(y_i), f_{i+1}^0(y_{i+1}), \dots, f_m^0(y_m) d\vec{y} \right| dz \\
 &\leq \frac{C}{|Q|} \int_Q \int_{R_i} \frac{|f_1^\infty(y_1) \cdots f_i^\infty(y_i) f_{i+1}^0(y_{i+1}) \cdots f_m^0(y_m)|}{(1 + |z - y_1| + \cdots + |z - y_m|)^N (|z - y_1| + \cdots + |z - y_m|)^{mn}} d\vec{y} dz \\
 &\leq \frac{C_N}{|Q|} \int_Q \left(\prod_{j=l+1}^m \int_{2Q} |f_j(y_j)| dy_j \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{(1 + |z - y_j|)^{\eta_j} |z - y_j|^{\frac{mn}{i}}} dy_j \right) dz \\
 &\leq C_N \left(\prod_{j=l+1}^m \int_{2Q} |f_j(y_j)| dy_j \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{(1 + |z - y_j|)^{\eta_j} |x - y_j|^{\frac{mn}{i}}} dy_j \right) \\
 &\leq C_N \prod_{j=l+1}^m \int_{2Q} |f_j(y_j)| dy_j \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{(1 + |z - y_j|)^{\eta_j} |x - y_j|^{\frac{mn}{i}}} dy_j \\
 &\leq C_N \prod_{j=l+1}^m M_{\varphi, \eta_j}(f_j)(x) (|Q|^{\frac{m-i}{i}})^i \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{(1 + |z - y_j|)^{\eta_j} |x - y_j|^{n + \frac{n(m-i)}{i}}} dy_j \\
 &\leq C_N \prod_{j=1}^m M_{\varphi, \eta_j}(f_j)(x),
 \end{aligned}$$

where we have used that $m > i$ and $N = m\eta$.

Applying Kolmogorov's estimate ([17]) to the term $T(\vec{f}^0) = T(f_1^0, \dots, f_m^0)(z)$, we have

$$\begin{aligned}
 \left(\frac{1}{|Q|} \int_Q |T(\vec{f}^0)(z)|^\delta dz \right)^{1/\delta} &\leq C \|T(\vec{f}^0)\|_{L^{1/m, \infty}(Q, \frac{dx}{|Q|})} \\
 &\leq C \prod_{j=1}^m \frac{1}{|Q|} \int_Q |f_j(z)| dz \leq C \prod_{j=1}^m M_{\varphi, \eta_j}(f_j)(x),
 \end{aligned}$$

since $T : L^1 \times \cdots \times L^1 \rightarrow L^{1/m, \infty}$; see [11].

Case 2. When $r \geq 1$. Similar to the proof of case 1. Taking $N = m\eta$, then

$$\begin{aligned}
 \left(\frac{1}{\varphi_\eta(Q)|Q|} \int_Q |T^\delta(\vec{f}^\infty)(z)| dz \right)^{1/\delta} &\leq \frac{C}{|Q|} \int_Q \int_{(\mathbb{R}^n)^m \setminus (2Q)^m} |K(z, \vec{w})| \prod_{j=1}^m |f_j(w_j)| d\vec{w} dy dz \\
 &\leq C_N \int_{(\mathbb{R}^n)^m \setminus (2Q)^m} \frac{\prod_{j=1}^m |f_j(w_j)| d\vec{w} dy}{(|y - w_1| + \cdots + |y - w_m|)^{mn+N}} \\
 &\leq C_N \prod_{j=1}^m \left(\int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(w_j)|}{|x - w_j|^{n+\frac{N}{m}}} d\vec{w} \right) \\
 &\leq C_N \prod_{j=1}^m \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(w_j)|}{|x - w_j|^{n+\frac{N}{m}}} dw_j \\
 &\leq C_N \prod_{j=1}^m M_{\varphi, \eta_j}(|f_j|)(x).
 \end{aligned}$$

The above computations gives the correct estimates for the first term in the right hand side of (4.12). To estimate the sum in the right hand side of (4.12) we distinguish between two kinds of term. One, in which at least one of the $k_j = \infty$, and one final term in which all the $k_j = 0$. A typical representative of the first kind of term is $T(f_1^\infty, \dots, f_i^\infty, f_{i+1}^0, \dots, f_m^0)(z)$. Using the notation $R_i = (\mathbb{R}^n \setminus 2Q)^i \times (2Q)^{m-i}$, by Minkowski's inequality, we have

$$\begin{aligned}
 &\left(\frac{1}{|Q|} \int_Q |T(f_1^\infty, \dots, f_i^\infty, f_{i+1}^0, \dots, f_m^0)(z)|^\delta dz \right)^{1/\delta} \\
 &\leq \frac{1}{|Q|} \int_Q |T(f_1^\infty, \dots, f_i^\infty, f_{i+1}^0, \dots, f_m^0)(z)| dz \\
 &\leq \frac{1}{|Q|} \int_Q \left| \int_{(\mathbb{R}^n)^m} k(x, \vec{y}) f_1^\infty(y_1), \dots, f_i^\infty(y_i), f_{i+1}^0(y_{i+1}), \dots, f_m^0(y_m) d\vec{y} \right| dz \\
 &\leq \frac{C_N}{|Q|} \int_Q \left(\int_{R_i} \frac{|f_1^\infty(y_1) \cdots f_i^\infty(y_i) f_{i+1}^0(y_{i+1}) \cdots f_m^0(y_m)|}{(1 + (|z - y_1| + \cdots + |z - y_m|))^N (|z - y_1| + \cdots + |z - y_m|)^{mn}} d\vec{y} \right) dz \\
 &\leq \frac{C_N}{|Q|} \int_Q \left(\prod_{j=l+1}^m \varphi(Q)^{-\eta_j} \int_{2Q} |f_j(y_j)| dy_j \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{|z - y_j|^{\frac{mn}{i} + \eta_j}} dy_j \right) dz \\
 &\leq C_N \left(\prod_{j=l+1}^m \int_{2Q} |f_j(y_j)| dy_j \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{|x - y_j|^{\frac{mn}{i} + \eta_j}} dy_j \right) \\
 &\leq C_N \prod_{j=l+1}^m \int_{2Q} |f_j(y_j)| dy_j \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{|x - y_j|^{\frac{mn}{i} + \eta_j}} dy_j \\
 &\leq C_N \prod_{j=l+1}^m M_{\varphi, \eta_j}(f_j)(x) (|Q|^{\frac{m-i}{i}})^i \prod_{j=1}^i \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_j(y_j)|}{|x - y_j|^{n + \frac{n(m-i)}{i} + \eta_j}} dy_j \\
 &\leq C_N \prod_{j=1}^m M_{\varphi, \eta_j}(f_j)(x),
 \end{aligned}$$

where we have used that $m > i$ and $N = m\eta$.

Applying Kolmogorov's estimate ([17]) to the term $T(\vec{f}^\delta) = T(f_1^0, \dots, f_m^0)(z)$, we have

$$\begin{aligned} \left(\frac{1}{\varphi_\eta(Q)|Q|} \int_Q |T(\vec{f}^\delta)(z)|^\delta dz \right)^{1/\delta} &\leq C \varphi_\eta(Q)^{-1} \|T(\vec{f}^\delta)\|_{L^{1/m, \infty}(Q, \frac{dx}{|Q|})} \\ &\leq C \prod_{j=1}^m \frac{1}{\varphi_{\eta_j}(Q)|Q|} \int_Q |f_j(z)| dz \\ &\leq C \prod_{j=1}^m M_{\varphi, \eta_j}(f_j)(x), \end{aligned}$$

since $T : L^1 \times \dots \times L^1 \rightarrow L^{1/m, \infty}$; see [11].

Hence, Lemma 4.3 is proved. \square

Applying Lemma 4.2 and 4.3, we show that for $1 < p < \infty$ and for all $\omega \in A_\infty^{1, \infty}$, for any $\eta_j > 0$, $j = 1, \dots, m$,

$$\|T(f_1, \dots, f_m)\|_{L^p(\omega)} \leq C \prod_{j=1}^m M_{\eta_j}(f_j)\|_{L^p(\omega)}. \quad (4.13)$$

The scalar estimate (3.3) just (4.13). But the vector-valued inequalities (3.4) and (3.5) are new and immediately yield the following result by applying Hölder's inequality and the norm inequalities for the maximal operator.

Theorem 4.4. *Let T be a multilinear pseudo-differential operator, $1 \leq p_1, \dots, p_m < \infty$, $1 < q_1, \dots, q_m < \infty$ and $0 < p, q < \infty$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}, \quad \frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}.$$

If $1 < p_1, \dots, p_m < \infty$ and $\omega \in A_{p_1}^{1, \infty} \cap \dots \cap A_{p_m}^{1, \infty}$, then

$$\| |T \vec{f}|_q \|_{L^p(\omega)} \leq C \prod_{j=1}^m \| |f|_{q_j} \|_{L^{p_j}(\omega)}. \quad (4.14)$$

If at least one $p_j = 1$ and $\omega \in A_1^{1, \infty}$, then

$$\| |T \vec{f}|_q \|_{L^{p, \infty}(\omega)} \leq C \prod_{j=1}^m \| |f|_{q_j} \|_{L^{p_j}(\omega)}. \quad (4.15)$$

Moreover, inequalities (4.14) and (4.15) hold with T^* in place of T , where T^* is the dual operator of T .

Remark. We will continue to study weighted inequalities for multilinear pseudo-differential operators in the forthcoming paper.

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